9

Trigonometry

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Three Angle Measure
Introduction to Trigonometry

LEARNING GOALS

In this lesson, you will:
• Explore trigonometric ratios as measurement conversions.
• Analyze the properties of similar right triangles.

KEY TERMS
• reference angle
• opposite side
• adjacent side

“I’ve been workin’ on the railroad, all the live long day.” Can you hear that tune in your head? Can you sing the first two notes? Those two notes are separated by an interval called a perfect fourth. The two notes of a perfect fourth vibrate at different frequencies, and these frequencies are always in the ratio 4 : 3. That is, the higher note of a perfect fourth vibrates about 1.33 times faster than the lower note.

What about the first two notes of “Frosty the Snowman”? Can you sing those? The interval between these notes is called a minor third. The ratio of the higher note frequency to lower note frequency in a minor third is 6 : 5.

All of the intervals in a musical scale are constructed according to specific frequency ratios.
**Problem 1 Convert to Trigonometry!**

You know that to convert between measurements you can multiply by a conversion ratio. For example, to determine the number of centimeters that is equivalent to 30 millimeters, you can multiply by $\frac{1 \text{ cm}}{10 \text{ mm}}$ because there are 10 millimeters in each centimeter:

$$30 \text{ mm} \times \frac{1 \text{ cm}}{10 \text{ mm}} = \frac{30 \text{ cm}}{10} = 3 \text{ cm}$$

In trigonometry, you use conversion ratios too. These ratios apply to right triangles.

Triangle $ABC$ shown is a 45°-45°-90° triangle.

1. Draw a vertical line segment, $DE$, connecting the hypotenuse of triangle $ABC$ with side $BC$. Label the endpoint of the vertical line segment along the hypotenuse as point $D$. Label the other endpoint as point $E$. 
2. Explain how you know that triangle \(ABC\) is similar to triangle \(DEC\).

3. Measure each of the sides of triangles \(ABC\) and \(DEC\) in millimeters. Record the approximate measurements.

You know that the hypotenuse of a right triangle is the side that is opposite the right angle. In trigonometry, the legs of a right triangle are often referred to as the opposite side and the adjacent side. These references are based on the angle of the triangle that you are looking at, which is called the reference angle. The opposite side is the side opposite the reference angle. The adjacent side is the side adjacent to the reference angle that is not the hypotenuse.

4. For triangles \(ABC\) and \(DEC\), identify the opposite side, adjacent side, and hypotenuse, using angle \(C\) as the reference angle.
5. Determine each side length ratio for triangles $ABC$ and $DEC$, using angle $C$ as the reference angle. Write your answers as decimals rounded to the nearest thousandth.

a. $\frac{\text{side opposite } \angle C}{\text{hypotenuse}}$

b. $\frac{\text{side adjacent to } \angle C}{\text{hypotenuse}}$

c. $\frac{\text{side opposite } \angle C}{\text{side adjacent to } \angle C}$
6. Draw two more vertical line segments, $FG$ and $HJ$, connecting the hypotenuse of triangle $ABC$ with side $BC$. Label the endpoints of the vertical line segments along the hypotenuse as points $F$ and $H$. Label the other endpoints as points $G$ and $J$.

a. Explain how you know that triangles $ABC$, $DEC$, $FGC$, and $HJC$ are all similar.

b. Measure each of the sides of the two new triangles you created. Record the side length measurements for all four triangles in the table.

<table>
<thead>
<tr>
<th>Triangle Name</th>
<th>Length of Side Opposite Angle C</th>
<th>Length of Side Adjacent to Angle C</th>
<th>Length of Hypotenuse</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle $ABC$</td>
<td></td>
<td></td>
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<tr>
<td>Triangle $DEC$</td>
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<tr>
<td>Triangle $FGC$</td>
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<td></td>
<td></td>
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<tr>
<td>Triangle $HJC$</td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

c. Determine each side length ratio for all four triangles using angle $C$ as the reference angle.

<table>
<thead>
<tr>
<th>Triangle Name</th>
<th>side opposite $\angle C$ hypotenuse</th>
<th>side adjacent to $\angle C$ hypotenuse</th>
<th>side opposite $\angle C$ side adjacent to $\angle C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle $ABC$</td>
<td></td>
<td></td>
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<tr>
<td>Triangle $DEC$</td>
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<td>Triangle $FGC$</td>
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<tr>
<td>Triangle $HJC$</td>
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</tbody>
</table>
7. Compare the side length ratios of all four triangles in the table. What do you notice?

8. Compare your measurements and ratios with those of your classmates. What do you notice?

9. Calculate the slope of the hypotenuse in each of the four triangles. Explain how you determined your answers.
Given the same reference angle measure, are each of the ratios you studied constant in similar right triangles? You can investigate this question by analyzing similar right triangles without side measurements.

Consider triangles $ABC$ and $DEC$ shown. They are both $45^\circ$-$45^\circ$-$90^\circ$ triangles.

Triangle $ABC$ is similar to triangle $DEC$ by the AA Similarity Theorem. This means that the ratios of the corresponding sides of the two triangles are equal.

\[
\frac{CE}{CB} = \frac{CD}{CA}
\]

Rewrite the proportion.

So, the ratio $\frac{\text{side adjacent to reference angle}}{\text{hypotenuse}}$ is constant in similar right triangles given the same reference angle measure.
10. Use triangle $ABC$ in the worked example with reference angle $C$ to verify that the ratios $\frac{\text{side opposite reference angle}}{\text{hypotenuse}}$ and $\frac{\text{side opposite reference angle}}{\text{side adjacent to reference angle}}$ are constant in similar right triangles. Show your work.

**PROBLEM 2 30°-60°-90°**

Triangle $PQR$ shown is a 30°-60°-90° triangle.

1. Draw three vertical line segments, $AB$, $CD$, and $EF$, connecting the hypotenuse of triangle $PQR$ with side $PR$. Label the endpoints of the vertical line segments along the hypotenuse as points $A$, $C$, and $E$. Label the other endpoints as points $B$, $D$, and $F$. 
2. Measure each of the sides of the four similar right triangles in millimeters. Record the side length measurements in the table.

<table>
<thead>
<tr>
<th>Triangle Name</th>
<th>Length of Side Opposite Angle $P$</th>
<th>Length of Side Adjacent to Angle $P$</th>
<th>Length of Hypotenuse</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle $PQR$</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Triangle $PEF$</td>
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<tr>
<td>Triangle $PCD$</td>
<td></td>
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<tr>
<td>Triangle $PAB$</td>
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</tr>
</tbody>
</table>

3. Determine each side length ratio for all four triangles using angle $P$ as the reference angle.

<table>
<thead>
<tr>
<th>Triangle Name</th>
<th>$\frac{\text{side opposite } \angle P}{\text{hypotenuse}}$</th>
<th>$\frac{\text{side adjacent to } \angle P}{\text{hypotenuse}}$</th>
<th>$\frac{\text{side opposite } \angle P}{\text{side adjacent to } \angle P}$</th>
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<tbody>
<tr>
<td>Triangle $PQR$</td>
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<tr>
<td>Triangle $PEF$</td>
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<tr>
<td>Triangle $PCD$</td>
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<td></td>
</tr>
<tr>
<td>Triangle $PAB$</td>
<td></td>
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</tbody>
</table>
4. Compare the side length ratios of all four triangles in the table. What do you notice?

5. What conclusions can you draw from Problem 1 and Problem 2 about the three ratios you studied in 45°-45°-90° triangles and 30°-60°-90° triangles?

6. Is each of the three ratios you studied in this lesson the same for any right triangles with congruent reference angles? Explain your reasoning.
7. Explain why Alicia is incorrect.

Alicia
The ratio \( \frac{BC}{AB} \) is equal to the ratio \( \frac{DC}{AD} \), because the ratio \( \frac{\text{side opposite } \angle A}{\text{hypotenuse}} \) is the same for both triangles ABC and ADC, given the reference angle A, which is 45°.

8. Is each of the three ratios you studied in this lesson the same for any triangles with congruent reference angles? Explain your reasoning.

The three ratios you worked with in this lesson are very important to trigonometry and have special names and properties. You will learn more about these ratios in the next several lessons.
Talk the Talk

1. As the reference angle measure increases, what happens to each ratio? Explain your reasoning.

   a. \[ \frac{\text{opposite}}{\text{hypotenuse}} \]

   b. \[ \frac{\text{adjacent}}{\text{hypotenuse}} \]

   c. \[ \frac{\text{opposite}}{\text{adjacent}} \]

Be prepared to share your solutions and methods.
The Tangent Ratio
Tangent Ratio, Cotangent Ratio, and Inverse Tangent

LEARNING GOALS
In this lesson, you will:
• Use the tangent ratio in a right triangle to solve for unknown side lengths.
• Use the cotangent ratio in a right triangle to solve for unknown side lengths.
• Relate the tangent ratio to the cotangent ratio.
• Use the inverse tangent in a right triangle to solve for unknown angle measures.

KEY TERMS
• rationalizing the denominator
• tangent (tan)
• cotangent (cot)
• inverse tangent

When we talk about "going off on a tangent" in everyday life, we are talking about touching on a topic and then veering off to talk about something completely unrelated.

“Tangent” in mathematics has a similar meaning—a tangent line is a straight line that touches a curve at just one point. In this lesson, though, you will see that tangent is a special kind of ratio in trigonometry.

It’s one you’ve already learned about!
PROBLEM 1 Wheelchair Ramps

The maximum incline for a safe wheelchair ramp should not exceed a ratio of 1 : 12. This means that every 1 unit of vertical rise requires 12 units of horizontal run. The maximum rise for any run is 30 inches. The ability to manage the incline of the ramp is related to both its steepness and its length.

Troy decides to build 2 ramps, each with the ratio 1 : 12.

1. The first ramp extends from the front yard to the front porch. The vertical rise from the yard to the porch is 2.5 feet.
   a. Draw a diagram of the ramp. Include the measurements for the vertical rise and horizontal run of the ramp.
   b. Calculate the length of the surface of the ramp.
2. The second ramp extends from the deck on the back of the house to the backyard. The vertical rise from the yard to the deck is 18 inches.
   a. Draw a diagram of the ramp. Include the measurements for the vertical rise and horizontal run of the ramp.
   b. Calculate the length of the surface of the ramp.

3. Compare the two ramps. Are the triangles similar? Explain your reasoning.

4. Compare and describe the angles of inclination of the two ramps.
PROBLEM 2  Slope and Right Triangles

In the wheelchair ramp problem, Troy used $1 : 12$ as the ratio of the rise of each ramp to the run of each ramp.

1. Describe the shape of each wheelchair ramp.

2. What does the ratio of the rise of the ramp to the run of the ramp represent?

3. Analyze the triangles shown.

4. a. Verify the triangles are similar. Explain your reasoning.

5. b. Calculate the ratio of the rise to the run for each triangle. How do the ratios compare?
A standard mathematical convention is to write fractions so that there are no irrational numbers in the denominator. **Rationalizing the denominator** is the process of rewriting a fraction so that no irrational numbers are in the denominator.

To rationalize the denominator of a fraction involving radicals, multiply the fraction by a form of 1 so that the product in the denominator includes a perfect square radicand. Then simplify, if possible.

Example 1:

\[
\frac{10}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}
\]

Example 2:

\[
\frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}
\]

4. Rewrite your answers in Question 3, part (b), by rationalizing the denominators. Show your work.
5. Analyze the triangles shown.

![Triangle Diagram]

a. Verify the triangles are similar. Explain your reasoning.

b. Calculate the ratio of the rise to the run for each triangle. How do the ratios compare?

6. What can you conclude about the ratios of the rise to the run in similar right triangles?

The tangent (tan) of an acute angle in a right triangle is the ratio of the length of the side that is opposite the reference angle to the length of the side that is adjacent to the reference angle. The expression “tan \( \angle A \)” means “the tangent of \( \angle A \).”

Consider \( \angle A \) in the right triangle shown.

The tangent ratio describes the relationship between \( \angle A \), the side opposite \( \angle A \), and the side adjacent to \( \angle A \).

\[
\tan \angle A = \frac{\text{length of side opposite } \angle A}{\text{length of side adjacent to } \angle A} = \frac{BC}{AC}
\]
7. Complete the ratio that represents the tangent of \( \angle B \).

\[
\tan B = \frac{\text{length of side opposite } \angle B}{\text{length of side adjacent to } \angle B} = \square \]

8. Determine the tangent values of all the acute angles in the right triangles from Questions 3 and 5.

9. What can you conclude about the tangent values of congruent angles in similar triangles?

10. Consider the tangent values in Question 8. In each triangle, compare \( \tan 30^\circ \) to \( \tan 60^\circ \). What do you notice? Why do you think this happens?
11. A proposed wheelchair ramp is shown.

   a. What information about the ramp is required to show that the ramp meets the safety rules?

   b. Write a decimal that represents the greatest value of the slope of a safe ramp.

   c. If you calculate the value of \( \tan 4^\circ \), how can you use this value to determine whether the ramp meets the safety rules?

   d. Use a calculator to determine the value of \( \tan 4^\circ \). Round your answer to the nearest hundredth.

   e. What is the ratio of the rise of the proposed ramp to its run? Is the ramp safe?
12. Another proposed wheelchair ramp is shown. What is the run of the ramp? If necessary, round your answer to the nearest inch.

13. Another proposed wheelchair ramp is shown. What is the rise of this ramp? If necessary, round your answer to the nearest inch.

14. If other ramps that have a 4° angle with different side measurements are drawn by extending or shortening the rise and run, will the tangent ratios always be equivalent? Explain your reasoning.
PROBLEM 3 Generally Speaking . . .

In the previous problems, you used the measure of an acute angle and the length of a side in a right triangle to determine the unknown length of another side.

Consider a right triangle with acute angles of unknown measures and sides of unknown lengths. Do you think the same relationships will be valid?

1. If one acute angle of a right triangle has a measure of $x$ degrees, what algebraic expression represents the measure of the second acute angle? Label this angle and explain your reasoning.

2. Suppose two right triangles are similar and each triangle contains an acute angle that measures $x$°, as shown.

   a. If the side opposite the acute angle measuring $x$° in the first triangle is of length $L_1$, what algebraic expression represents the length of the side opposite the acute angle measuring $x$° in the second triangle?

   b. If the side opposite the acute angle measuring $90 - x$° in the first triangle is of length $L_2$, what algebraic expression represents the length of the side opposite the acute angle measuring $90 - x$° in the second triangle?
3. Write an expression to represent the tangent of the angle measuring \( x^\circ \) in the first triangle.

4. Write an expression to represent the tangent of the angle measuring \( x^\circ \) in the second triangle.

5. Write a proportion to represent the relationship between the two triangles in terms of the tangents of the angle measuring \( x^\circ \).

**PROBLEM 4  Cotangent Ratio**

The cotangent (cot) of an acute angle in a right triangle is the ratio of the length of the side that is adjacent to the angle to the length of the side that is opposite the angle. The expression “cot \( A \)” means “the cotangent of \( \angle A \).”

![Diagram of a right triangle with labels A, B, and C]

1. Complete the ratio that represents the cotangent of \( \angle A \).
   \[ \cot A = \frac{\text{length of side adjacent to } \angle A}{\text{length of side opposite } \angle A} \]

2. Prove algebraically that the cotangent of \( A = \frac{1}{\tan A} \).
3. As the measure of an acute angle increases, the tangent value of the acute angle increases. Explain the behavior of the cotangent value of an acute angle as the measure of the acute angle increases.

4. A ski slope at Snowy Valley has an average angle of elevation of 21°.

![Diagram of a ski slope with angles and sides labeled]

a. Calculate the vertical height of the ski slope \( x \) using the cotangent ratio.

b. Calculate the vertical height of the ski slope \( x \) using the tangent ratio.

c. Which ratio did you prefer to use when calculating the value of \( x \)? Explain your reasoning.
5. Are all right triangles that contain a 21° angle similar? Why or why not?

6. If other right triangles containing a 21° angle with different side measurements are drawn by extending or shortening the rise and run, will the cotangent ratios always be equivalent?

**PROBLEM 5 Inverse Tangent**

The inverse tangent (or arc tangent) of \( x \) is defined as the measure of an acute angle whose tangent is \( x \). If you know the length of any two sides of a right triangle, it is possible to compute the measure of either acute angle by using the inverse tangent, or the \( \tan^{-1} \) button on a graphing calculator.

In right triangle \( ABC \), if \( \tan A = x \), then \( \tan^{-1} x = m \angle A \).

1. Consider triangle \( ABC \) shown.
   a. If \( \tan A = \frac{15}{10} \) then calculate \( \tan^{-1}\left(\frac{15}{10}\right) \) to determine \( m \angle A \).

b. Determine the ratio for \( \tan B \), and then use \( \tan^{-1}(\tan B) \) to calculate \( m \angle B \).
c. Add $m\angle A$ and $m\angle B$. Does your sum make sense in terms of the angle measures of a triangle?

2. Calculate $m\angle E$.

3. Movable bridges are designed to open waterways for large boats and barges. When the bridge moves, all vehicle traffic stops. The maximum height of the open bridge deck of the movable bridge shown is 37 feet above the water surface. The waterway width is 85 feet. Calculate the angle measure formed by the movement of the bridge.
Many mountainous areas have road signs like this sign that refer to the percentage grade for the road. An 8% grade, for example, means that the altitude changes by 8 feet for each 100 feet of horizontal distance.

1. To determine the percentage grade that should be put on a road sign where the angle of elevation of the road is 9°, what function would be most helpful?

2. To determine the angle of elevation of a road with a percentage grade of 7%, what function would be most helpful?

3. Determine the angle of elevation of a road with a percentage grade of 6%.

4. Determine the percentage road grade that should be put on a road sign where the angle of elevation is 10°.

5. Does the image in the sign accurately represent an 8% grade? Explain how you can determine the answer.

6. What is the approximate angle of elevation that is actually shown in the sign?

Be prepared to share your solutions and methods.
The Sine Ratio
Sine Ratio, Cosecant Ratio, and Inverse Sine

LEARNING GOALS

In this lesson, you will:
• Use the sine ratio in a right triangle to solve for unknown side lengths.
• Use the cosecant ratio in a right triangle to solve for unknown side lengths.
• Relate the sine ratio to the cosecant ratio.
• Use the inverse sine in a right triangle to solve for unknown angle measures.

KEY TERMS
• sine (sin)
• cosecant (csc)
• inverse sine

Measuring angles on paper is easy when you have a protractor. But what about measuring angles in the real world? You can build an astrolabe (pronounced uh-STRAW-luh-bee) to help you.

Copy and cut out the astrolabe shown (without the straw). You will probably want to glue the astrolabe to cardboard or heavy paper before cutting it out.

Cut a drinking straw to match the length of the one shown. Tape the straw to the edge labeled “Place straw on this side” so that it rests on the astrolabe as shown.

Poke a hole through the black dot shown and pass a string through this hole. Knot the string or tape it so that it stays in place.

Finally, tie a weight to the end of the string. You’re ready to go!
FORE!

Each golf club in a set of clubs is designed to cause the ball to travel different distances and different heights. One design element of a golf club is the angle of the club face.

You can draw a right triangle that is formed by the club face angle. The right triangles formed by different club face angles are shown.

1. How do you think the club face angle affects the path of the ball?

2. For each club face angle, write the ratio of the side length opposite the given acute angle to the length of the hypotenuse. Write your answers as decimals rounded to the nearest hundredth.
3. What happens to this ratio as the angle measure gets larger?

The sine (sin) of an acute angle in a right triangle is the ratio of the length of the side that is opposite the angle to the length of the hypotenuse. The expression “sin A” means “the sine of \( \angle A \)."

4. Complete the ratio that represents the sine of \( \angle A \).

\[
\sin A = \frac{\text{length of side opposite } \angle A}{\text{length of hypotenuse}} = \frac{\text{ }}{\text{}}
\]

5. For each triangle in Problem 1, calculate the sine value of the club face angle. Then calculate the sine value of the other acute angle. Round your answers to the nearest hundredth.

6. What do the sine values of the angles in Question 5 all have in common?

7. Jun says that the sine value of every acute angle is less than 1. Is Jun correct? Explain your reasoning.
8. What happens to the sine values of an angle as the measure of the angle increases?

9. Use the right triangles shown to calculate the values of \( \sin 30^\circ \), \( \sin 45^\circ \), and \( \sin 60^\circ \).

10. A golf club has a club face angle \( A \) for which \( \sin A \approx 0.45 \). Estimate the measure of \( \angle A \). Use a calculator to verify your answer.
The cosecant (csc) of an acute angle in a right triangle is the ratio of the length of the hypotenuse to the length of the side that is opposite the angle. The expression “csc \( A \)” means “the cosecant of \( \angle A \).”

1. Complete the ratio that represents the cosecant of \( \angle A \).

\[
\text{csc} \ A = \frac{\text{length of hypotenuse}}{\text{length of side opposite } \angle A}
\]

2. Prove algebraically that the cosecant of \( A = \frac{1}{\sin A} \).

3. As the measure of an acute angle increases, the sine value of the acute angle increases. Explain the behavior of the cosecant value of an acute angle as the measure of the acute angle increases.
**PROBLEM 3 Inverse Sine**

The inverse sine (or arc sine) of $x$ is defined as the measure of an acute angle whose sine is $x$. If you know the length of any two sides of a right triangle, it is possible to calculate the measure of either acute angle by using the inverse sine, or $\sin^{-1}$ button on a graphing calculator.

In right triangle $ABC$, if $\sin A = x$, then $\sin^{-1} x = m\angle A$.

1. In right triangle $ABC$, if $\sin A = \frac{2}{5}$, calculate $\sin^{-1} \left( \frac{2}{5} \right)$ to determine $m\angle A$.

2. Determine the ratio for $\sin B$, and then use $\sin^{-1}(\sin B)$ to calculate $m\angle B$.

3. Calculate $m\angle B$. 

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The graphical representation of the triangle is shown with the sides labeled as follows:

- In the first part, the triangle is labeled with sides $AB = 25$, $AC = 11$, and $BC$.
- In the second part, the triangle is labeled with sides $AB = 12$, $AC = 5$, and $BC$.
4. The movable bridge shown is called a double-leaf Bascule bridge. It has a counterweight that continuously balances the bridge deck, or “leaf,” throughout the entire upward swing, providing an open waterway for boat traffic. The counterweights on double-leaf bridges are usually located below the bridge decks. The length of one leaf, or deck, is 42 feet. The maximum height of an open leaf is 30 feet. Calculate the measure of the angle formed by the movement of the bridge.

5. The Leaning Tower of Pisa is a tourist attraction in Italy. It was built on unstable land and, as a result, it really does lean! The height of the tower is approximately 55.86 meters from the ground on the low side and 56.7 meters from the ground on the high side. The top of the tower is displaced horizontally 3.9 meters as shown.

Determine the angle at which the tower leans.
Talk the Talk

Match each diagram with the appropriate situation and identify the trigonometric function that would be most helpful in answering each question. Then calculate the unknown measurement.

Diagram A

Diagram B

Diagram C

Diagram D

Diagram E
1. A building is 80 feet high. An observer stands an unknown distance away from the building and, looking up to the top of the building, notes that the angle of elevation is 52°. Determine the distance from the base of the building to the observer.

2. A building is 80 feet high. An observer stands 62.5 feet away from the building and looking up to the top of the building he ponders the measure of the angle of elevation. Determine the measure of the angle of elevation.

3. An observer stands an unknown distance away from a building and looking up to the top of the building notes that the angle of elevation is 52°. He also knows the distance from where he is standing to the top of the building is 101.52 feet. Determine the height of the building.
4. A building is 80 feet high. An observer stands an unknown distance away from the building and looking up to the top of the building he ponders the measure of the angle of elevation. He also knows the distance from where he is standing to the top of the building is 101.52 feet. Determine the measure of the angle of elevation.

5. A building is 80 feet high. An observer stands an unknown distance away from the building and looking up to the top of the building notes that the angle of elevation is 52°. Determine the distance from the observer to the top of the building.

Be prepared to share your solutions and methods.
The Cosine Ratio

Cosine Ratio, Secant Ratio, and Inverse Cosine

**LEARNING GOALS**

In this lesson, you will:

- Use the cosine ratio in a right triangle to solve for unknown side lengths.
- Use the secant ratio in a right triangle to solve for unknown side lengths.
- Relate the cosine ratio to the secant ratio.
- Use the inverse cosine in a right triangle to solve for unknown angle measures.

**KEY TERMS**

- cosine (cos)
- secant (sec)
- inverse cosine

The applications of trigonometry are tremendous. Engineering, acoustics, architecture, physics . . . you name it, they probably use it.

One important application of trigonometry can be found in finding things—specifically, where you are in the world using GPS, or the Global Positioning System. This system employs about two dozen satellites communicating with a receiver on Earth. The receiver talks to 4 satellites at the same time, uses trigonometry to calculate the information received, and then tells you where on Earth you are.
Problem 1 Making a Tower Stable

A “guy wire” is used to provide stability to tall structures like radio towers. Guy wires are attached near the top of a tower and are attached to the ground.

A guy wire and its tower form a right triangle. It is important that all guy wires form congruent triangles so that the tension on each wire is the same.

1. Each triangle shown represents the triangle formed by a tower and guy wire. The angle formed by the wire and the ground is given in each triangle.

For each acute angle formed by the wire and the ground, write the ratio of the length of the side adjacent to the angle to the length of the hypotenuse. Write your answers as decimals rounded to the nearest hundredth if necessary.

The cosine (cos) of an acute angle in a right triangle is the ratio of the length of the side that is adjacent to the angle to the length of the hypotenuse. The expression “cos A” means “the cosine of ∠A.”

2. Complete the ratio to represent the cosine of ∠A.

\[ \cos A = \frac{\text{length of side adjacent to } \angle A}{\text{length of hypotenuse}} = \]
3. For each triangle in Question 1, calculate the cosine value of the angle made by the guy wire and the ground. Then calculate the cosine value of the other acute angle. Round your answers to the nearest hundredth if necessary.

4. What do the cosine values of the angles in Question 3 all have in common?

5. Is the cosine value of every acute angle less than 1? Explain your reasoning.

6. What happens to the cosine value of an angle as the measure of the angle increases?

7. Use the right triangles shown to calculate the values of \( \cos 30^\circ \), \( \cos 45^\circ \), and \( \cos 60^\circ \). Show all your work.
8. A guy wire is 600 feet long and forms a 55° angle with the ground. First, draw a diagram of this situation. Then, calculate the number of feet from the tower's base to where the wire is attached to the ground.

9. Firemen are climbing a 65´ ladder to the top of a 56´ building. Calculate the distance from the bottom of the ladder to the base of the building, and use the cosine ratio to compute the measure of the angle formed where the ladder touches the top of the building.
10. For the triangle shown, calculate the values of $\sin 30^\circ$, $\cos 30^\circ$, and $\tan 30^\circ$.

11. Calculate the value of $\frac{\sin 30^\circ}{\cos 30^\circ}$.

12. What do you notice about the value of $\frac{\sin 30^\circ}{\cos 30^\circ}$?

13. Do you think that the relationship between the sine, cosine, and tangent values of an angle is true for any angle? Explain your reasoning.
The secant (sec) of an acute angle in a right triangle is the ratio of the length of the hypotenuse to the length of the side that is adjacent to the angle. The expression “sec \( A \)” means “the secant of \( \angle A \).”

1. Complete the ratio to represent the secant of \( \angle A \).

\[
\sec A = \frac{\text{length of hypotenuse}}{\text{length of side adjacent to } \angle A} = \frac{5}{}\]

2. Prove algebraically that the secant of \( A = \frac{1}{\cos A} \).

3. As the measure of an acute angle increases, the cosine value of the acute angle decreases. Explain the behavior of the secant value of an acute angle as the acute angle increases.

4. If there is no “sec” button on your graphing calculator, how can you compute the secant?
5. The diagram shows the measurements for the ski slope in the first lesson.

\[ \begin{align*}
\text{x} & \quad \text{4689.2 ft} \\
21^\circ & \\
\end{align*} \]

a. Use the cosine function to determine the length of the slope.

b. Use the secant function to determine the length of the slope.
The inverse cosine (or arc cosine) of \( x \) is defined as the measure of an acute angle whose cosine is \( x \). If you know the length of any two sides of a right triangle, it is possible to compute the measure of either acute angle by using the inverse cosine, or \( \cos^{-1} \) button on a graphing calculator.

In right triangle \( ABC \), if \( \cos A = x \), then \( \cos^{-1} x = m \angle A \).

1. In right triangle \( ABC \), if \( \cos A = \frac{2}{7} \), calculate \( \cos^{-1}\left(\frac{2}{7}\right) \) to determine \( m \angle A \).

2. Determine the ratio for \( \cos B \), and then use \( \cos^{-1}(\cos B) \) to calculate \( m \angle B \).

3. Calculate \( m \angle B \).
4. A typical cable-stayed bridge is a continuous girder with one or more towers erected above piers in the middle of the span. From these towers, cables stretch down diagonally (usually to both sides) and support the girder. Tension and compression are calculated into the design of this type of suspension bridge.

One cable is 95 feet. The span on the deck of the bridge from that cable to the girder is 80 feet. Calculate the angle formed by the deck and the cable.
5. Diane is training for a charity bicycle marathon. She leaves her house at noon and heads due west, biking at an average rate of 4 miles per hour. At 3 PM she changes course to N 65°W as shown. Determine the bike's distance from Diane's home at 5 PM.
Talk the Talk

1. Match each trigonometric function with the appropriate abbreviation.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Sine</td>
<td>A. $\cos^{-1}$</td>
</tr>
<tr>
<td>2. Cosine</td>
<td>B. $\cot$</td>
</tr>
<tr>
<td>3. Tangent</td>
<td>C. $\csc$</td>
</tr>
<tr>
<td>4. Cosecant</td>
<td>D. $\tan^{-1}$</td>
</tr>
<tr>
<td>5. Secant</td>
<td>E. $\sin^{-1}$</td>
</tr>
<tr>
<td>6. Cotangent</td>
<td>F. $\cos$</td>
</tr>
<tr>
<td>7. Arc tan</td>
<td>G. $\sin$</td>
</tr>
<tr>
<td>8. Arc sin</td>
<td>H. $\tan$</td>
</tr>
<tr>
<td>9. Arc cos</td>
<td>I. $\sec$</td>
</tr>
</tbody>
</table>

2. Match each trigonometric function with the appropriate description.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Sin</td>
<td>A. $\frac{\text{hypotenuse}}{\text{opposite}}$</td>
</tr>
<tr>
<td>2. Cos</td>
<td>B. $\frac{\text{hypotenuse}}{\text{adjacent}}$</td>
</tr>
<tr>
<td>3. Tan</td>
<td>C. $\frac{\text{opposite}}{\text{hypotenuse}}$</td>
</tr>
<tr>
<td>4. Csc</td>
<td>D. $\frac{\text{opposite}}{\text{adjacent}}$</td>
</tr>
<tr>
<td>5. Sec</td>
<td>E. $\frac{\text{adjacent}}{\text{opposite}}$</td>
</tr>
<tr>
<td>6. Cot</td>
<td>F. $\frac{\text{adjacent}}{\text{hypotenuse}}$</td>
</tr>
</tbody>
</table>
3. Given the known information and the solution requirement, determine which function can be used to solve each situation.

<table>
<thead>
<tr>
<th>Known Information</th>
<th>Solution Requirement</th>
<th>Function Used</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Hypotenuse</td>
<td>Measure of reference angle</td>
<td></td>
</tr>
<tr>
<td>• Opposite</td>
<td></td>
<td>Hypotenuse</td>
</tr>
<tr>
<td>• Acute angle measure</td>
<td></td>
<td>Adjacent</td>
</tr>
<tr>
<td>• Opposite</td>
<td>Measure of reference angle</td>
<td></td>
</tr>
<tr>
<td>• Adjacent</td>
<td>Opposite</td>
<td></td>
</tr>
<tr>
<td>• Hypotenuse</td>
<td></td>
<td>Measure of reference angle</td>
</tr>
<tr>
<td>• Acute angle measure</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Be prepared to share your solutions and methods.

Sometimes it is helpful to use a mnemonic device to transform information into a form that you can easily remember. Can you come up with a mnemonic for the trig ratios?
We Complement Each Other!
Complement Angle Relationships

LEARNING GOALS
In this lesson, you will:
• Explore complement angle relationships in a right triangle.
• Solve problems using complement angle relationships.

You’ve worked with complements before, remember? Two angles whose measures add up to 90 degrees are called complements.

In right triangles, complements are pretty easy to locate. The two angles whose measures are not 90 degrees must be complements. In trigonometry, complements are easy to identify by name, too. The prefix “co-” in front of trigonometric ratio names stands for “complement.”
PROBLEM 1 Angles Are Very Complementary!

Consider triangle ABC with right angle C. Angles A and B are complementary angles because the sum of their measures is equal to 90 degrees. The trigonometric ratios also have complementary relationships.

1. Use triangle ABC to answer each question.

1. Use triangle ABC to answer each question.

- **a.** Compare the ratios that represent \( \sin \angle A \) and \( \cos \angle B \).

- **b.** Compare the ratios that represent \( \sin \angle B \) and \( \cos \angle A \).

- **c.** Compare the ratios that represent \( \csc \angle A \) and \( \sec \angle B \).

- **d.** Compare the ratios that represent \( \tan \angle A \) and \( \cot \angle B \).
2. Which two functions are described by the ratio \( \frac{c}{b} \)?

3. Which two functions are described by the ratio \( \frac{b}{a} \)?

4. Use your answers to Questions 1 through 3 to complete the table.

<table>
<thead>
<tr>
<th>Reference Angle</th>
<th>sin</th>
<th>cos</th>
<th>tan</th>
<th>csc</th>
<th>sec</th>
<th>cot</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( B )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. Summarize the relationship between the trigonometric functions of complementary angles.
PROBLEM 2 Using Complements

1. Given: $\sin 30^\circ = 0.5$
   Use the Pythagorean Theorem and your knowledge of complementary functions to complete the chart. Include both a ratio and its decimal equivalent for each function.

<table>
<thead>
<tr>
<th>Reference Angle</th>
<th>sin</th>
<th>cos</th>
<th>tan</th>
<th>csc</th>
<th>sec</th>
<th>cot</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>A</td>
<td>C</td>
<td>B</td>
</tr>
<tr>
<td>60°</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>A</td>
<td>C</td>
<td>B</td>
</tr>
</tbody>
</table>

2. Given: $\sin 45^\circ = 0.707$
   Use the Pythagorean Theorem and your knowledge of complementary functions to complete the chart. Include both a ratio and its decimal equivalence for each function.

<table>
<thead>
<tr>
<th>Reference Angle</th>
<th>sin</th>
<th>cos</th>
<th>tan</th>
<th>csc</th>
<th>sec</th>
<th>cot</th>
</tr>
</thead>
<tbody>
<tr>
<td>45°</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>A</td>
<td>C</td>
<td>B</td>
</tr>
</tbody>
</table>
3. Trafalgar Square is a tourist attraction located in London, England, United Kingdom. The name commemorates the Battle of Trafalgar (1805), a British naval victory of the Napoleonic Wars over France.

\[
\text{a. Use a trigonometric function to solve for the dimensions of Trafalgar Square.}
\]

\[
\text{b. Use the Pythagorean Theorem to solve for the dimensions of Trafalgar Square.}
\]
1. At an altitude of 1,000 feet, a balloonist measures the angle of depression from the balloon to the landing zone. The measure of that angle is 15°. How far is the balloon from the landing zone?
2. To measure the width of the Grand Canyon, a surveyor stands at a point on the North Rim of the canyon and measures the angle of depression to a point directly across on the South Rim of the canyon.

At the surveyor’s position on the North Rim, the Grand Canyon is 7,256 feet above sea level. The point on the South Rim, directly across, is 6,159 feet above sea level. Sketch a diagram of the situation and determine the width of the Grand Canyon at the surveyor’s position.
3. An aircraft uses radar to spot another aircraft 8,000 feet away at a 12° angle of depression. Sketch the situation and determine the vertical and horizontal separation between the two aircraft.
4. When a space shuttle returns from a mission, the angle of its descent to the ground from the final 10,000 feet above the ground is between $17^\circ$ and $19^\circ$ with the horizontal. Sketch a diagram of the situation and determine the maximum and minimum horizontal distances between the landing site and where the descent begins.

Be prepared to share your solutions and methods.
Time to Derive!

Deriving the Triangle Area Formula, the Law of Sines, and the Law of Cosines

In this lesson, you will:

- Derive the formula for the area of a triangle using the sine function.
- Derive the Law of Sines.
- Derive the Law of Cosines.

Suppose you want to measure the height of a tree. You are 100 feet from the tree, and the angle from your feet to the top of the tree is 33 degrees. However, the tree isn’t growing straight up from the ground. It leans a little bit toward you. The tree is actually growing out of the ground at an 83 degree angle. How tall is the tree?

Once you finish this lesson, see if you can answer this question.
PROBLEM 1 Deriving Another Version of the Area Formula

Whether you are determining the area of a right triangle, solving for the unknown side lengths of a right triangle, or solving for the unknown angle measurements in a right triangle, the solution paths are fairly straightforward. You can use what you learned previously, such as the area formula for a triangle, the Pythagorean Theorem, and the Triangle Sum Theorem.

Solving for unknown measurements of sides or angles of a triangle becomes more involved if the given triangle is not a right triangle.

Consider triangle ABC as shown.

Use of the area formula requires the height of the triangle, which is not given.

Use of the Pythagorean Theorem requires the triangle to be a right triangle, which it is not.

Use of the Triangle Sum Theorem requires the measures of two angles of the triangle, which are not given.

In this lesson, you will explore how trigonometric ratios are useful when determining the area of any triangle, solving for unknown lengths of sides of any triangle, and solving for unknown measures of angles in any triangle.

1. Analyze triangle ABC.
   a. Write the formula for the area of triangle ABC in terms of b and h.
b. Write the ratio that represents \( \sin C \) and solve for the height, \( h \).

c. Rewrite the formula you wrote for the area of triangle \( ABC \) in Question 1 by substituting the expression for the value of \( h \) in Question 2.

The area formula, \( A = \frac{1}{2} ab \sin C \), can be used to determine the area of any triangle if you know the lengths of two sides and the measure of the included angle.

2. Use a trigonometric ratio to determine the area of the triangle.
PROBLEM 2 Deriving the Law of Sines

You have used the trigonometric ratios to solve for unknown side length and angle measures in right triangles. Let’s explore relationships between side lengths and angle measures in any triangle.

1. Analyze triangle $ABC$.
   
   a. Write a ratio to represent $\sin A$, and then solve for the height, $h$.

   b. Write a ratio that represents $\sin C$, and then solve for the height, $h$.

   c. What can you conclude about the relationship between $c \cdot \sin A$ and $a \cdot \sin C$?

   d. Express $c \cdot \sin A = a \cdot \sin C$ as a proportion.

   e. Write a ratio that represents $\sin B$, and then solve for the height, $k$.

   f. Write a ratio that represents $\sin C$, and then solve for the height, $k$. 

---

Chapter 9 Trigonometry
g. What can you conclude about the relationship between $c \cdot \sin B$ and $b \cdot \sin C$?

h. Express $c \cdot \sin B = b \cdot \sin C$ as a proportion.

i. Derive the Law of Sines by combining the proportions formed in parts (d) and (h).

The Law of Sines, or \( \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \), can be used to determine the unknown side lengths or the unknown angle measures in any triangle.

2. Use the Law of Sines to determine the measure of \( \angle B \).
**PROBLEM 3** Deriving the Law of Cosines

The Law of Sines is one relationship between the side lengths and angle measures of any triangle. Another relationship is called the **Law of Cosines**.

1. Analyze triangle ABC.

![Diagram of triangle ABC with labels a, b, c, h, x, A, B, C]

a. Write a ratio that represents \( \sin A \), and then solve for the height, \( h \).

b. Write a ratio that represents \( \cos A \), and then solve for \( x \).

c. Solve for \( a^2 \) using the Pythagorean Theorem.

d. Substitute the expressions for \( h \) and \( x \) into the equation in part (c).

e. Use the equation you wrote in part (d) to solve for \( a^2 \).

Think about the identity \( \sin^2 A + \cos^2 A = 1 \).
2. Repeat the steps in Question 1 to solve for $b^2$. 
3. Repeat the steps in Question 1 to solve for $c^2$.

The Law of Cosines, or

\[
\begin{align*}
    a^2 &= c^2 + b^2 - 2bc \cdot \cos A \\
    b^2 &= a^2 + c^2 - 2ac \cdot \cos B \\
    c^2 &= a^2 + b^2 - 2ab \cdot \cos C
\end{align*}
\]

can be used to determine the unknown lengths of sides or the unknown measures of angles in any triangle.
4. Why is the Pythagorean Theorem considered to be a special case of the Law of Cosines?

**PROBLEM 4 Applying Yourself!**

A surveyor was hired to determine the approximate length of a proposed tunnel which will be necessary to complete a new highway. A mountain stretches from point A to point B as shown. The surveyor stands at point C and measures the distance from where she is standing to both points A and B, then measures the angle formed between these two distances.

1. Use the surveyor’s measurements to determine the length of the proposed tunnel.

![Diagram of a surveyor measuring distances and angles]
2. A nature lover decides to use geometry to determine if she can swim across a river. She locates two points, A and B, along one side of the river and determines the distance between these points is 250 meters. She then spots a point C on the other side of the river and measures the angles formed using point C to point A and then point C to point B. She determines the measure of the angle whose vertex is located at point A to be 35° and the angle whose vertex is located at point B to be 127° as shown.

How did she determine the distance across the river from point B to point C and what is that distance?
3. A typical direct flight from Pittsburgh, Pennsylvania, to New York City is approximately 368 miles. A pilot alters the course of his aircraft 33° for 85 miles to avoid a storm and then turns the aircraft heading straight for New York City, as shown.

![Diagram showing flight path with storm avoidance]

a. How many additional miles did the aircraft travel to avoid the storm?

b. If a commercial jet burns an average of 11.875 liters per kilometer, and the cost of jet fuel is $3.16 per gallon, this alteration in route due to the storm cost the airline company how much money?
Talk the Talk

1. When is it appropriate to use the Law of Sines?

2. When is it appropriate to use the Law of Cosines?

Be prepared to share your solutions and methods.
Chapter 9 Summary

KEY TERMS
- reference angle (9.1)
- opposite side (9.1)
- adjacent side (9.1)
- rationalizing the denominator (9.2)
- tangent (tan) (9.2)
- cotangent (cot) (9.2)
- inverse tangent (9.2)
- sine (sin) (9.3)
- cosecant (csc) (9.3)
- inverse sine (9.3)
- cosine (cos) (9.4)
- secant (sec) (9.4)
- inverse cosine (9.4)
- Law of Sines (9.6)
- Law of Cosines (9.6)

9.1 Analyzing the Properties of Similar Right Triangles

In similar triangles, the ratio \( \frac{\text{opposite}}{\text{hypotenuse}} \) is equal for corresponding reference angles.

In similar triangles, the ratio \( \frac{\text{adjacent}}{\text{hypotenuse}} \) is equal for corresponding reference angles.

In similar triangles, the ratio \( \frac{\text{opposite}}{\text{adjacent}} \) is equal for corresponding reference angles.

Example

Right triangles \( ABC \) and \( ADE \) are similar. Consider angle \( A \) as the reference angle.

\( \frac{\text{opposite}}{\text{hypotenuse}} \) ratios:
- triangle \( ABC \): \( \frac{2.0}{2.5} = 0.8 \)
- triangle \( ADE \): \( \frac{6.0}{7.5} = 0.8 \)

\( \frac{\text{adjacent}}{\text{hypotenuse}} \) ratios:
- triangle \( ABC \): \( \frac{1.5}{2.5} = 0.6 \)
- triangle \( ADE \): \( \frac{4.5}{7.5} = 0.6 \)

\( \frac{\text{opposite}}{\text{adjacent}} \) ratios:
- triangle \( ABC \): \( \frac{0.8}{0.6} \approx 1.33 \)
- triangle \( ADE \): \( \frac{0.8}{0.6} \approx 1.33 \)
## 9.2 Using the Tangent Ratio

The tangent (tan) of an acute angle in a right triangle is the ratio of the length of the side that is opposite the angle to the length of the side that is adjacent to the angle. You can use the tangent of an angle to determine the length of a leg in a right triangle when you know the measure of an acute angle and the length of the other leg.

### Example

![Diagram of a right triangle with angle A = 42°, side AC = 1.5 ft, and side AB = x.]

\[
tan 42° = \frac{x}{1.5}
\]

\[
1.5(tan 42°) = x
\]

\[
x ≈ 1.35 \text{ ft}
\]

## 9.2 Using the Cotangent Ratio

The cotangent (cot) of an acute angle in a right triangle is the ratio of the length of the side that is adjacent to the angle to the length of the side that is opposite the angle. You can use the cotangent of an angle to determine the length of a leg in a right triangle when you know the measure of an acute angle and the length of the other leg.

### Example

![Diagram of a right triangle with angle GFH = 55°, side GF = 6 m, and side FH = a.]

\[
cot 55° = \frac{a}{6}
\]

\[
6(cot 55°) = a
\]

\[
6\left(\frac{1}{tan 55°}\right) = a
\]

\[
a ≈ 4.20 \text{ m}
\]
9.2 **Using the Inverse Tangent**

The inverse tangent, or arc tangent, of \( x \) is defined as the measure of an acute angle whose tangent is \( x \). You can use the inverse tangent to calculate the measure of either acute angle in a right triangle when you know the lengths of both legs.

**Example**

\[
\begin{align*}
\theta_X &= \tan^{-1}\left(\frac{5}{14}\right) \\
&\approx 19.65^\circ \\
\theta_Y &= \tan^{-1}\left(\frac{14}{5}\right) \\
&\approx 70.35^\circ
\end{align*}
\]

9.2 **Solving Problems Using the Tangent Ratio**

An angle of elevation is the angle above a horizontal. You can use trigonometric ratios to solve problems involving angles of elevation.

**Example**

Mitchell is standing on the ground 14 feet from a building and he is looking up at the top of the building. The angle of elevation that his line of sight makes with the horizontal is 65°. His eyes are 5.2 feet from the ground. To calculate the height of the building, first draw a diagram of the situation. Then write and solve an equation involving a trigonometric ratio.

\[
\begin{align*}
\tan 65^\circ &= \frac{x}{14} \\
14(\tan 65^\circ) &= x \\
&\approx 30 \text{ ft} \\
x + 5.2 &= 30 + 5.2 = 35.2 \text{ ft}
\end{align*}
\]

The building is about 35.2 feet tall.
9.3 Using the Sine Ratio

The sine (sin) of an acute angle in a right triangle is the ratio of the length of the side that is opposite the angle to the length of the hypotenuse. You can use the sine of an angle to determine the length of a leg in a right triangle when you know the measure of the angle opposite the leg and the length of the hypotenuse. You can also use the sine of an angle to determine the length of the hypotenuse when you know the measure of an acute angle and the length of the leg opposite the angle.

Example

![Diagram of a right triangle with sides labeled S, T, and R, and an angle of 61 degrees at R.]

\[
\sin 61^\circ = \frac{y}{18} \\
18(\sin 61^\circ) = y \\
x \approx 15.74 \text{ cm}
\]

9.3 Using the Cosecant Ratio

The cosecant (csc) of an acute angle in a right triangle is the ratio of the length of the hypotenuse to the length of the side that is opposite the angle. You can use the cosecant of an angle to determine the length of a leg in a right triangle when you know the measure of the angle opposite the leg and the length of the hypotenuse. You can also use the cosecant of an angle to determine the length of the hypotenuse when you know the measure of an acute angle and the length of the leg opposite the angle.

Example

![Diagram of a right triangle with sides labeled P, Q, and R, and an angle of 33 degrees at R.]

\[
csc 33^\circ = \frac{a}{10} \\
10(csc 33^\circ) = a \\
10\left(\frac{1}{\sin 33^\circ}\right) = a \\
a \approx 18.36 \text{ in.}
\]
9.3 Using the Inverse Sine

The inverse sine, or arc sine, of \( x \) is defined as the measure of an acute angle whose sine is \( x \). You can use the inverse sine to calculate the measure of an acute angle in a right triangle when you know the length of the leg opposite the angle and the length of the hypotenuse.

Example

![Diagram of a right triangle with sides labeled J, K, L, and measurements 44 yd, 18 yd, and \( m\angle J \) labeled.]

\[
m\angle J = \sin^{-1}\left(\frac{18}{44}\right) \approx 24.15^\circ
\]

9.4 Using the Cosine Ratio

The cosine (\( \cos \)) of an acute angle in a right triangle is the ratio of the length of the side that is adjacent to the angle to the length of the hypotenuse. You can use the cosine of an angle to determine the length of a leg in a right triangle when you know the measure of the angle adjacent to the leg and the length of the hypotenuse. You can also use the cosine of an angle to determine the length of the hypotenuse when you know the measure of an acute angle and the length of the leg adjacent to the angle.

Example

![Diagram of a right triangle with sides labeled W, X, V, and measurements 26°, 9 mm, and \( c \) labeled.]

\[
\cos 26^\circ = \frac{c}{9} \\
9(\cos 26^\circ) = c \\
c \approx 8.09 \text{ mm}
\]
9.4 Using the Secant Ratio

The secant (sec) of an acute angle in a right triangle is the ratio of the length of the hypotenuse to the length of the side that is adjacent to the angle. You can use the secant of an angle to determine the length of a leg in a right triangle when you know the measure of the angle adjacent to the leg and the length of the hypotenuse. You can also use the secant of an angle to determine the length of the hypotenuse when you know the measure of an acute angle and the length of the leg adjacent to the angle.

Example

\[ \sec 48^\circ = \frac{x}{25} \]

\[ 25(\sec 48^\circ) = x \]

\[ 25\left(\frac{1}{\cos 48^\circ}\right) = x \]

\[ x = 37.36 \text{ ft} \]

9.4 Using the Inverse Cosine

The inverse cosine, or arc cosine, of \( x \) is defined as the measure of an acute angle whose cosine is \( x \). You can use the inverse cosine to calculate the measure of an acute angle in a right triangle when you know the length of the leg adjacent to the angle and the length of the hypotenuse.

Example

\[ m\angle P = \cos^{-1}\left(\frac{7}{16}\right) \approx 64.06^\circ \]
Exploring Complementary Angle Relationships in a Right Triangle

The two acute angles of a right triangle are complementary angles because the sum of their measures is 90 degrees. The trigonometric ratios also have complementary relationships:

The sine of an acute angle is equal to the cosine of its complement.

The tangent of an acute angle is equal to the cotangent of its complement.

The secant of an acute angle is equal to the cosecant of its complement.

Example

Angle $A$ and angle $B$ are complementary angles.

\[
\sin \angle A = \frac{a}{c} \quad \quad \quad \quad \quad \sin \angle B = \frac{b}{c} \\
\cos \angle B = \frac{a}{c} \quad \quad \quad \quad \quad \cos \angle A = \frac{b}{c} \\
\sin \angle A \text{ and } \cos \angle B \text{ are the same ratio.} \quad \quad \quad \quad \quad \sin \angle B \text{ and } \cos \angle A \text{ are the same ratio.}
\]

\[
\tan \angle A = \frac{a}{b} \quad \quad \quad \quad \quad \tan \angle B = \frac{b}{a} \\
\cot \angle B = \frac{a}{b} \quad \quad \quad \quad \quad \cot \angle A = \frac{b}{a} \\
\tan \angle A \text{ and } \cot \angle B \text{ are the same ratio.} \quad \quad \quad \quad \quad \tan \angle B \text{ and } \cot \angle A \text{ are the same ratio.}
\]

\[
\sec \angle A = \frac{c}{b} \quad \quad \quad \quad \quad \sec \angle B = \frac{c}{a} \\
\csc \angle B = \frac{c}{b} \quad \quad \quad \quad \quad \csc \angle A = \frac{c}{a} \\
\sec \angle A \text{ and } \csc \angle B \text{ are the same ratio.} \quad \quad \quad \quad \quad \sec \angle B \text{ and } \csc \angle A \text{ are the same ratio.}
\]
Solving Problems Using Complementary Angle Relationships

An angle of elevation is the angle below a horizontal. You can use trigonometric ratios to solve problems involving angles of depression.

Example

You are standing on a cliff and you see a house below you. You are 50 feet above the house. The angle of depression that your line of sight makes with the horizontal is 33°. To calculate the horizontal distance \( x \) you are from the house, first draw a diagram of the situation. Then write and solve an equation involving a trigonometric ratio.

\[
\tan 57^\circ = \frac{x}{50}
\]

\[
50(\tan 57^\circ) = x
\]

\[
x \approx 77 \text{ ft}
\]

You are about 77 feet from the house.

Deriving and Using a Formula for the Area of a Triangle

You can calculate the area of any triangle if you know the lengths of two sides and the measure of the included angle.

Area formula:

For any triangle \( ABC \), \( A = \frac{1}{2}ab\sin C \).

Example

\[
A = \frac{1}{2}df(\sin E)
\]

\[
A = \frac{1}{2}(10)(12)(\sin 85^\circ)
\]

\[
A \approx 59.8 \text{ square centimeters}
\]
### 9.6 Deriving and Using the Law of Sines

You can use the Law of Sines when

- you know the lengths of two sides of a triangle and the measure of an angle opposite one of those sides, and you want to know the measure of the angle opposite the other known side.

or

- you know the measures of two angles of a triangle and the length of a side opposite one of those angles, and you want to know the length of the side opposite the other known angle.

**Law of Sines:**

For any triangle $ABC$, \[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.
\]

**Example**

![Diagram of a triangle with sides labeled 6 cm, 7 cm, and 7 cm, and angles labeled 72° and 7 cm.]

\[
\begin{align*}
\sin A &= \frac{\sin 72^\circ}{6} \\
\sin B &= \frac{\sin C}{7} \\
\sin 72^\circ &= \sin C \\
6\sin 72^\circ &= 7\sin C \\
\sin C &= \frac{6\sin 72^\circ}{7} = 0.815 \\
C &\approx 54.6^\circ
\end{align*}
\]
9.6 Deriving and Using the Law of Cosines

You can use the Law of Cosines when

- You know the lengths of all three sides of a triangle and you want to solve for the measure of any of the angles

or

- You know the lengths of two sides of a triangle and the measure of the included angle, and you want to solve for the length of the third side.

Law of Cosines

For any triangle $ABC$,

\[ a^2 = b^2 + c^2 - 2bc \cos A \]
\[ b^2 = a^2 + c^2 - 2ac \cos B \]
\[ c^2 = a^2 + b^2 - 2ab \cos C. \]

Example

\[ a^2 = b^2 + c^2 - 2bc \cos A \]
\[ a^2 = 6.5^2 + 6.0^2 - 2(6.5)(6.0)(\cos 45^\circ) \]
\[ a^2 = 42.25 + 36 - 78(\cos 45^\circ) \]
\[ a^2 \approx 23.10 \]
\[ a \approx 4.8 \text{ inches} \]