Looking Ahead to Chapter 10

Focus
In Chapter 10, you will learn about polynomials, including how to add, subtract, multiply, and divide polynomials. You will also learn about polynomial and rational functions.

Chapter Warm-up

Answer these questions to help you review skills that you will need in Chapter 10.

Solve each two-step equation.

1. \(5x + 3 = 2x - 12\)
2. \(4x = 7 - 3x\)
3. \(\frac{x}{5} + 3 = \frac{x}{10}\)

Simplify each expression.

4. \((3^3)^5\)
5. \(x^7 \cdot x^{-5}(x^2)^{-3}\)
6. \(\left(\frac{xy^5}{y^3}\right)^{-1}\)

Read the problem scenario below.

You have just planted a flower bed in a 3-foot wide by 5-foot long rectangular section of your yard. After planting some flowers, you decide that you would like to add to the width of your garden so that there is more space to plant flowers. Let \(x\) represent the amount added to the length of your garden.

7. Represent the area of your expanded garden with an expression in two ways by using the distributive property.

8. What is the area of your garden if you expand the width by 4 feet?

Key Terms

- polynomial ■ p. 454
- term ■ p. 454
- coefficient ■ p. 454
- degree of a term ■ p. 454
- degree of the polynomial ■ p. 454
- monomial ■ p. 455
- binomial ■ p. 455
- trinomial ■ p. 455
- standard form ■ p. 455
- Vertical Line Test ■ p. 457
- combine like terms ■ p. 460
- distributive property ■ p. 460
- area model ■ p. 464
- divisor ■ p. 468
- dividend ■ p. 468
- remainder ■ p. 468
- FOIL pattern ■ p. 472
- square of a binomial sum ■ p. 473
- square of a binomial difference ■ p. 475
- rational expression ■ p. 483
- domain ■ p. 483
- restrict the domain ■ p. 485
Stained glass windows have been used as decoration in homes, religious buildings, and other buildings since the 7th century. In Lesson 10.4, you will find the area of a stained glass window.
Objectives
In this lesson, you will:
- Identify terms and coefficients of polynomials.
- Classify polynomials by the number of terms.
- Classify polynomials by degree.
- Write polynomials in standard form.
- Use the Vertical Line Test to determine whether equations are functions.

Key Terms
- polynomial
- term
- coefficient
- degree
- monomial
- binomial
- trinomial
- standard form
- Vertical Line Test

Take Note
The vertical motion model is
\[ y = -16t^2 + vt + h, \]
where \( t \) is the time in seconds that the object has been moving, \( v \) is the initial velocity (speed) in feet per second of the object, \( h \) is the initial height in feet of the object, and \( y \) is the height in feet of the object at time \( t \).

SCENARIO  On a calm day, you and a friend are tossing water balloons in a field trying to hit a boulder in the field. The balloons travel in a path that is in the shape of a parabola.

Problem 1  Ready, Set, Launch

A. On your first throw, the balloon leaves your hand 3 feet above the ground at a velocity of 20 feet per second. Use what you learned in Chapter 8 about vertical motion models to write an equation that gives the height of the balloon in terms of time.

B. What is the height of the balloon after one second? Show your work and use a complete sentence to explain your answer.

C. What is the height of the balloon after two seconds? Show your work and use a complete sentence in your answer.

D. On your second throw, the balloon leaves your hand at ground level at a velocity of 30 feet per second. Write an equation that gives the height of the balloon in terms of time.

E. What is the height of the balloon after one second? Show your work and use a complete sentence in your answer.

F. What is the height of the balloon after two seconds? Show your work and use a complete sentence in your answer.
1. How are your models the same? Use a complete sentence in your answer.

How are your models different? Use a complete sentence in your answer.

2. Just the Math: Polynomials The expressions 
\[-16t^2 + 20t + 3 \text{ and } -16t^2 + 30t\] that model the heights of the balloons are polynomials. Each expression is a polynomial because it is a sum of products of the form \(ax^k\), where \(a\) is a real number and \(k\) is a whole number. Each product is a term and the number being multiplied by a power is a coefficient. For each of the polynomials above, name the terms and coefficients. Use complete sentences in your answer.

3. For each of your polynomial models, what is the greatest exponent? Use a complete sentence in your answer.

The degree of a term in a polynomial is the exponent of the term. The greatest exponent in a polynomial determines the degree of the polynomial. For instance, in the polynomial 
\[4x + 3\], the greatest exponent is 1, so the degree of the polynomial is 1. What is the degree of each of your polynomial models? Use a complete sentence in your answer.

4. What kind of expression is a polynomial of degree 0? Give an example and use a complete sentence to explain your reasoning.

What kind of expression is a polynomial of degree 1? Give an example and use a complete sentence to explain.
Investigate Problem 1

What kind of expression is a polynomial of degree 2? Give an example and use a complete sentence to explain your reasoning.

A polynomial of degree 3 is called a cubic polynomial. Write an example of a cubic polynomial.

5. For each of your polynomial models in parts (A) and (D), find the number of terms in the model. Use a complete sentence in your answer.

Polynomials with only one term are monomials. Polynomials with exactly two terms are binomials. Polynomials with exactly three terms are trinomials. Classify each polynomial model in parts (A) and (D) by its number of terms. Use a complete sentence in your answer.

6. Give an example of a monomial of degree 3.

Give an example of a trinomial of degree 5.

7. Just the Math: Standard Form of a Polynomial

Later in this chapter, we will be adding, subtracting, multiplying, and dividing polynomials. To make this process easier, it is helpful to write polynomials in standard form. A polynomial is written in standard form by writing the terms in descending order, starting with the term with the greatest degree and ending with the term with the least degree. Write each polynomial in standard form.

\[
6 + 5x \quad 7 - x^2
\]

\[
4 + 3x + 4x^2 \quad 4 + 3x^2 + 9x - x^3
\]

\[
5 - 6x^4 \quad x^6 - 4x^3 + 16
\]
Investigate Problem 1

8. Determine whether each algebraic expression is a polynomial. If the expression is a polynomial, classify it by its degree and number of terms. If the expression is not a polynomial, use a complete sentence to explain why it is not a polynomial.

$4x^2 + 3x - 1$

$3x^{-2} + 4x - 1$

$4x^6 + 1$

$10 - x^4$

$2 \sqrt{x} + 3x - 4$

$25$

Problem 2

The Balloon’s Path

It is now your friend’s turn to throw water balloons at the boulder.

A. In your friend’s first throw, the balloon leaves her hand 2 feet above the ground at a velocity of 36 feet per second. Write an equation that gives the height of the balloon in terms of time.

B. Complete the table of values that shows the height of the balloon in terms of time.

<table>
<thead>
<tr>
<th>Quantity Name</th>
<th>Unit</th>
<th>Expression</th>
<th>Time</th>
<th>Height</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.0</td>
<td></td>
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<tr>
<td></td>
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<td></td>
<td>0.5</td>
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<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3.0</td>
<td></td>
</tr>
</tbody>
</table>
Problem 2 The Balloon’s Path

C. Create a graph of the model to see the path of the balloon on the grid below. First, choose your bounds and intervals. Be sure to label your graph clearly.

Investigate Problem 2

1. Is the equation that you wrote in part (A) a function? How do you know? Use a complete sentence in your answer.

2. Just the Math: Vertical Line Test You can use a graph to determine whether an equation is a function. The Vertical Line Test states that an equation is a function if you can pass a vertical line through any part of the graph of the equation and the line intersects the graph, at most, one time. Consider your graph in part (C). Does your graph pass the Vertical Line Test?
3. Consider each equation and its graph. Use the Vertical Line Test to determine whether the equation is a function.

**Take Note**
Whenever you see the share with the class icon, your group should prepare a short presentation to share with the class that describes how you solved the problem. Be prepared to ask questions during other groups’ presentations and to answer questions during your presentation.
Objectives
In this lesson, you will:
■ Add polynomials.
■ Subtract polynomials.

Key Terms
■ combine like terms
■ distributive property
■ add
■ subtract

SCENARIO A friend of yours loves baseball and plans to visit every major league baseball park in the country. The major league is made up of two divisions, the National League and the American League.

Problem 1 Batter Up

Your friend recently read an article that stated that the attendance at National League baseball games from 1990 through 2001 can be modeled by the function
\[ y = -86,584x^3 + 1,592,363x^2 - 5,692,368x + 24,488,926 \]
where \( x \) is the number of years since 1990 and \( y \) is the number of people who attended. The article also stated that the attendance at American League baseball games from 1990 through 2001 can be modeled by the function
\[ y = -56,554x^3 + 1,075,426x^2 - 4,806,571x + 30,280,751 \]
where \( x \) is the number of years since 1990 and \( y \) is the number of people who attended.

A. Find the attendance for each league in 1995. Show your work and use a complete sentence in your answer.

B. Find the total attendance at all games in major league baseball in 1995. Show your work and use a complete sentence in your answer.

How did you find your answer to part (B)? Use a complete sentence in your answer.
Investigate Problem 1

1. How would you write a function that you could use to find the total attendance at all major league baseball games? Use a complete sentence in your answer.

2. You can add or subtract two polynomials by combining like terms. What are the like terms in the attendance polynomial models?

Find the sum of each group of like terms. What is the model for total attendance at all major league baseball games?

Problem 1  Batter Up

C. Find the attendance for each league in 2000. Show your work and use a complete sentence in your answer.

D. Find the total attendance at all games in major league baseball in 2000. Show your work and use a complete sentence in your answer.

How did you find your answer to part (D)? Use a complete sentence in your answer.

Take Note

Remember that you use a distributive property to combine like terms:

\[ 4x + 10x = x(4 + 10) \]
\[ = x(14) \]
\[ = 14x \]
Investigate Problem 1

3. Use your model to find the total attendance in 1995 and 2000. Show your work and use a complete sentence in your answer.

How do these answers compare to those in Problem 1 parts (B) and (D)? Use a complete sentence in your answer.

When is it useful for you to find the sum of two functions first, and then evaluate the function rather than evaluating each function separately and then finding the sum? Use a complete sentence in your answer.

4. You can also write a function that shows how many more people attended National League games each year than attended American League games. Describe how you can find this function. Use a complete sentence in your answer.

Complete the statement below that gives the function described above.

\[ y = \ldots \]

Because you are subtracting one function from another function, you must subtract each term of the second function from the first function. To do this, use the distributive property to distribute the negative sign to each term in the second function.

\[ y = -86,584x^3 + 1,592,363x^2 - 5,692,368x + 24,488,926 \]
Investigate Problem 1

Now combine like terms and simplify.

5. What was the difference in attendance between the leagues in 1998? Which league had more people attending games in 1998? How do you know? Show your work and use complete sentences in your answer.

What was the difference in attendance between the leagues in 1992? Which league had more people attending games in 1992? How do you know? Show your work and use a complete sentence in your answer.

6. Simplify each expression by finding the sum or difference. Show your work.

\[(4x^2 + 3x - 5) + (x^2 - 8x + 4)\]

\[(2x^2 + 3x - 4) - (2x^2 + 5x - 6)\]

\[(4x^3 + 5x - 2) + (7x^2 - 8x + 9)\]

\[(9x^4 - 5) - (8x^4 - 2x^3 + x)\]
Objectives
In this lesson, you will:
- Use an area model to multiply polynomials.
- Use distributive properties to multiply polynomials.
- Use long division to divide polynomials.

Key Terms
- area model
- distributive property
- divisor
- dividend
- remainder

SCENARIO
Your high school is trying to decide whether to offer more foreign language classes. The guidance counselor finds a model for the percent of high school students that were in a foreign language class in high school during the years from 1985 to 2000, which is

\[ y = -0.0005t^2 + 0.02t + 0.23 \]

where \( t \) represents the number of years since 1980 and \( y \) is the percent (in decimal form) of all high school students who were in a foreign language class. The guidance counselor also finds that the total number of students in high school during the years from 1985 to 2000 can be modeled by

\[ y = -3t^3 + 134t^2 - 1786t + 18,398 \]

where \( t \) represents the number of years since 1980 and \( y \) is the number of students in thousands.

Problem 1 Learning a Foreign Language

A. What percent of high school students were in a foreign language class in 1990? Show your work and use a complete sentence in your answer.

B. Find the total number of students in high school in 1990. Show your work and use a complete sentence in your answer.

C. In 1990, how many students were in a foreign language class? Show your work and use a complete sentence in your answer.

D. Use complete sentences to explain how you found your answer to part (C).
1. Use a complete sentence to explain how you can use the functions given in Problem 1 to write a function that gives the number of students that were in a foreign language class during the years from 1985 to 2000.

2. **Just the Math: Area Model** You can use an area model to multiply two polynomials. Suppose that you want to find the product of $3x$ and $4x + 1$. Consider these polynomials to be the length and width of a rectangle as shown below. The area of the rectangle is the product of the polynomials.

What is the area of each square in the model? Use a complete sentence in your answer.

What is the area of each small rectangle in the model? Use a complete sentence in your answer.

What is the area of the entire rectangle? Show your work and use a complete sentence in your answer.

3. **Just the Math: Multiplying Polynomials** To multiply two polynomials, you need to use the distributive properties of multiplication and the properties of exponents. The number of times that you need to use a distributive property depends on the number of terms in the polynomials. For instance, to multiply the polynomials $3x$ and $4x + 1$ from Question 2 above, you only need to use the distributive property once. Use the distributive property to complete the first step below.

$$(3x)(4x + 1) = (\_\_\_\_)(\_\_\_\_) + (\_\_\_\_)(\_\_\_\_)$$
Investigate Problem 1

To simplify each product, remember that multiplication is commutative (you can multiply numbers in any order) and use the properties of exponents that you learned in Chapter 9. For instance, \((2x)(5x) = 2(5)(x)(x) = 10x^2\). Complete the product of \(3x\) and \(4x + 1\) below.

\[(3x)(4x + 1) = (3x)(4x) + (3x)(1) = \underline{} + \underline{}\]

To multiply the polynomials \(x + 1\) and \(x^2 - 3x + 2\), you need to use the distributive property twice. First, use a distributive property to multiply each term of \(x + 1\) by the polynomial \(x^2 - 3x + 2\). Complete the first step below.

\[(x + 1)(x^2 - 3x + 2) = (x)(\underline{} \underline{}) + (1)(\underline{} \underline{})\]

Now, distribute \(x\) to each term of \(x^2 - 3x + 2\) and distribute 1 to each term of \(x^2 - 3x + 2\). Complete the second step below:

\[(x + 1)(x^2 - 3x + 2) = (x)(x^2) + (\underline{}) (\underline{}) + (\underline{}) (\underline{}) + (\underline{}) (\underline{}) + (\underline{}) (\underline{}) + (\underline{}) (\underline{})\]

Now multiply and collect like terms. Show your work and write your answer as a polynomial in standard form.

4. What do you notice about the products of the terms in Question 3 when you used a distributive property the second time? Use a complete sentence in your answer.

5. Find each product. Show all your work.

\[
2x(x + 3) \quad \quad 5x^2(7x - 1)
\]

\[
(x + 1)(x + 3) \quad \quad (x^2 - 4)(2x + 3)
\]

\[
(x - 5)(x^2 + 3x + 1)
\]
6. Use multiplication to find the function that gives the number of students in thousands that were in a foreign language class during the years from 1985 to 2000.

\[-0.0005t^2 + 0.02t + 0.23)(-3t^3 + 134t^2 - 1786t + 18,398)\]

First, distribute each term of the polynomial for the percent to each term of the polynomial for the total number of students.

\[-0.0005t^2(-3t^3 + 134t^2 - 1786t + 18,398) +
0.02t(-3t^3 + 134t^2 - 1786t + 18,398) +
0.23(-3t^3 + 134t^2 - 1786t + 18,398)\]

Complete the steps below. Show your work.

\[-0.0005t^2(-3t^3 + 134t^2 - 1786t + 18,398)\]

= \______________

\______________

= \______________

\[-0.0005t^2(-3t^3 + 134t^2 - 1786t + 18,398)\]

= \______________

\______________

= \______________

\[0.02t(-3t^3 + 134t^2 - 1786t + 18,398)\]

= \______________

\______________

= \______________

\[0.23(-3t^3 + 134t^2 - 1786t + 18,398)\]

= \______________

\______________

= \______________

Now combine like terms and simplify. Show your work.

What is your function?

7. How many students were in a foreign language class in 2000? Show all your work and use a complete sentence in your answer.
Problem 2

Spanish, Anyone?

A model for the number of high school students that were in a Spanish class in high school during the years from 1985 to 2000 is

\[ y = 4t^2 + 14t + 2137 \]

where \( t \) represents the number of years since 1980 and \( y \) is the number of high school students in thousands that were in a Spanish class.

A. Find the number of students that were in a Spanish class in 1990. Show your work and use a complete sentence in your answer.

B. Find the total number of students in high school in 1990. You can use the function

\[ y = -3t^3 + 134t^2 - 1786t + 18,398 \]

from Problem 1. Show your work and use a complete sentence in your answer.

C. In 1990, what percent of all high school students were in a Spanish class? Show your work and use a complete sentence in your answer.

D. Use complete sentences to explain how you found your answer to part (C).

Investigate Problem 2

1. Use a complete sentence to explain how you can use the functions given in Problem 2 to write a function that gives the percent of students that were in a Spanish class during the years from 1985 to 2000.
Investigate Problem 2

2. You can divide two polynomials just like you divide numbers by using long division. For instance, consider the quotient of $10x^3 + 13x^2 + 2x - 4$ and $2x + 1$ shown near the bottom of the page. Begin by finding the quotient of the first terms. In this case, what is $10x^3$ divided by $2x$?

This is the first term of the quotient. Write this term in the quotient below. Then multiply this expression by the polynomial $2x + 1$. Subtract the result from $10x^3 + 13x^2$ to get $8x^2$ (see Step 1).

Bring down the next term, $2x$, from the dividend. You need to do this because the product of $2x + 1$ and the second term of the quotient will have two terms. Now, we have to divide $8x^2$ by the $2x$ in the divisor. What is the result? This is the second term of the quotient.

Write this term in the quotient below. Then multiply this expression by the polynomial $2x + 1$. Subtract the result from $8x^2 + 2x$ (see Step 2 below).

Bring down the last term, $-4$, from the dividend. Finally, we have to divide $-2x$ by $2x$. What is the result? This is the last term of the quotient.

Write this term in the quotient below. Then multiply this expression by the polynomial $2x + 1$. Subtract the result from $-2x - 4$ (see Step 3 below).

Because the difference $-3$ is a term whose degree is less than the degree of the divisor ($2x$), we are done. This number $-3$ is the remainder. Write the remainder over the divisor below.
Investigate Problem 2

You can check your answer by multiplying the divisor and quotient and adding the remainder:

<table>
<thead>
<tr>
<th>Divisor</th>
<th>Quotient</th>
<th>Remainder</th>
<th>Dividend</th>
</tr>
</thead>
<tbody>
<tr>
<td>((2x + 1)(\underline{\quad}) + (\underline{\quad}))</td>
<td>(\underline{\quad})</td>
<td>(\underline{\quad})</td>
<td>(10x^3 + 13x^2 + 2x - 4)</td>
</tr>
<tr>
<td>(\underline{\quad})</td>
<td>(\underline{\quad})</td>
<td>(\underline{\quad})</td>
<td>(10x^3 + 13x^2 + 2x - 4)</td>
</tr>
<tr>
<td>(\underline{\quad})</td>
<td>(\underline{\quad})</td>
<td>(\underline{\quad})</td>
<td>(10x^3 + 13x^2 + 2x - 4)</td>
</tr>
</tbody>
</table>

3. Find each quotient. Show all your work.

\((x^2 + 6x + 5) \div (x + 1) = \underline{\quad}\)

Take Note

When the dividend and divisor polynomials are in standard form and the difference between consecutive exponents is greater than 1, you should write in a placeholder for the missing terms. For instance, you should write \(x^3 + 4x - 1\) as \(x^3 + 0x^2 + 4x - 1\).

\((2x^2 + 5x - 12) \div (2x - 3) = \underline{\quad}\)
Investigate Problem 2

\[(x^3 + 2x - 6) \div (x + 1) = \]________

4. Complete the division problem below to find the function that represents the percentage of the total number of high school students that are enrolled in a Spanish class.

\[(-3t^3 + 134t^2 - 1786.00t + 18,398) \div (4t^2 + 14t + 2137)\]

\[= \underline{ } + \underline{ } + \frac{4t^2 + 14t + 2137}{4t^2 + 14t + 2137}\]

\[4t^2 + 14t + 2137)\underline{-3t^3 + 134t^2 - 1786.00t + 18,398}\]
Objectives
In this lesson, you will:
■ Use the FOIL pattern to multiply binomials.
■ Use formulas to find special products.

Key Terms
■ FOIL pattern
■ square of a binomial sum
■ square of a binomial difference

SCENARIO
An artist creates designs in stained glass.
One of the artist’s more popular designs is a rectangle in which the ratio of the length to the width of the stained glass is 7:5.
The artist also surrounds the stained glass with a two-inch wide carved wooden frame.

Problem 1  Stained Glass Design

A. Complete the diagram below that models the stained-glass design in its frame.

B. Write an expression for the total length of the stained-glass design with its frame.

C. Write an expression for the total width of the stained-glass design with its frame.

D. Write a function for the area of the stained-glass design with the frame.

E. Use a distributive property to simplify the function in part (D). Show your work.

F. What type of function is in part (E)? Use a complete sentence in your answer.
1. What is the value of writing the function in standard form in part (E)? Show your work and use a complete sentence in your answer.

2. In part (E), you found the product of two binomials by using the distributive property. Consider the product after two uses of the distributive property:

\[(2x + 3)(x + 8) = (2x + 3)(x) + (2x + 3)(8)\]  First use of distributive property

\[(2x + 3)(x + 8) = 2x(x) + 2x(8) + 3(x) + 3(8)\]  Second use of distributive property

You can quickly find the product of two binomials by remembering the word FOIL. FOIL stands for multiplying the first terms, the outer terms, the inner terms, and the last terms. Then the products are added together as shown above. Finish finding the product of \(2x + 3\) and \(x + 8\).

3. Use the FOIL pattern to find each product. Show your work.

\[(x + 2)(x + 9)\hspace{1cm}(x - 1)(x + 7)\]

\[(x - 3)(x - 6)\hspace{1cm}(4x + 3)(x + 5)\]

\[(3x - 4)(x + 2)\hspace{1cm}(5x + 1)(3x - 2)\]
Problem 2  New Stained Glass Design

The stained glass artist is working on a new design that is square with a carved wood frame that is three inches wide.

A. Draw a diagram that models the stained-glass design with its frame. Let \( x \) be the width of the stained glass without the frame.

B. Write a function for the area of the stained-glass design with its frame.

C. Use the FOIL pattern to simplify your function in part (B). Show all your work.

D. Suppose that the artist changes the width of the frame to five inches wide. Write and simplify a new function for the area of the stained glass with the frame. Show all your work.

Investigate Problem 2

1. Use complete sentences to describe the similarities and differences between the functions in parts (C) and (D).

2. Just the Math: Square of a Binomial Sum  In Problem 2, you wrote and simplified the square of a binomial. In each simplified product, the middle term related to the product of the linear term and the constant term in the binomial. For instance, the square of \((x + 3)\) is \(x^2 + 6x + 9\). How is the middle term \(6x\) related to \(x\) and \(3\)? Use a complete sentence to explain.
Investigate Problem 2

Does this relationship hold true for the square of the binomials in parts (C) and (D)? Use complete sentences to explain.

In \((x + 3)(x + 3)\), which is \(x^2 + 6x + 9\), how is the last term 9 related to the constant term 3? Use a complete sentence to explain.

Does this relationship hold true for the square of the binomials in parts (C) and (D)? Use complete sentences to explain.

Complete the formula below for simplifying the square of a binomial sum of the form \((a + b)^2\). Then use the FOIL pattern to check your answer.

\[
(a + b)^2 = \square^2 + 2\square + \square^2
\]

FOIL:

3. Use the FOIL pattern to find the products \((x - 4)^2\) and \((4x - 3)^2\). Show your work.
4. Complete the formula for simplifying the **square of a binomial difference** of the form \((a - b)^2\). Then use the FOIL pattern to show that your formula works.
\[
(a - b)^2 = \square^2 - 2\square + \square^2
\]
FOIL:

5. Use the FOIL pattern to find the products \((x - 3)(x + 3)\) and \((2x - 5)(2x + 5)\). Show your work.

6. Write a formula for simplifying the product of a sum and a difference of the form \((a - b)(a + b)\). Then use the FOIL pattern to show that your formula works.

FOIL:

7. Find each product. Show your work.
\[
(x + 5)(x + 3) \\
(x - 6)^2 \\
(3x + 4)(x - 2) \\
(2x + 1)^2 \\
(x + 9)(x - 9) \\
(3x + 2)(3x - 2)
\]
**Objectives**
In this lesson, you will:
- Factor a polynomial by factoring out a common factor.
- Factor a polynomial of the form $x^2 + bx + c$.
- Factor a polynomial of the form $ax^2 + bx + c$.

**Key Terms**
- factor
- linear factor
- trinomial
- FOIL pattern

**SCENARIO**  In your science class, you have to make a presentation on a “feat of engineering.” You have decided to make your presentation on suspension bridges, such as the one shown below. During the research for the presentation, you have discovered that the cables that are used to carry the weight on a bridge form the shape of a parabola. Because you have recently studied parabolas in your math class, you decide to use what you know about parabolas in your presentation.

**Problem 1  Modeling the Cable Shape**
You have modeled the main cable’s shape of a particular suspension bridge by using the function $y = 0.002x^2 - 0.4x + 96$, where $x$ is the distance in meters from the left tower and $y$ is the height in meters of the cable above the high water level.

A. Write and simplify an equation that you can use to find the cable’s horizontal distance from the left tower at 96 meters above the high water level. Show your work.

B. How can you solve the equation in part (A)? Use a complete sentence in your answer.

C. Solve the equation in part (A). Show all your work.
Investigate Problem 1

1. In the last lesson, you multiplied polynomials. The product of what kinds of polynomials gives you a quadratic (second-degree) polynomial? Use a complete sentence in your answer.

Consider your equation again from part (A):

\[0.002x^2 - 0.4x = 0\]

Let’s simplify this equation. Because 0.002 evenly divides 0.4, divide each side by 0.002. What is the new equation?

If we could write the polynomial \(x^2 - 200x\) as the product of two linear polynomials, how would this help us be able to solve the equation \(x^2 - 200x = 0\)? Use complete sentences in your answer.

What expression do the terms of \(x^2 - 200x\) have in common?

Rewrite your equation so that each term is written as a product with this common expression as a factor.

The product of what two linear expressions gives you \(x^2 - 200x\)?

So now you have the following.

\[x^2 - 200x = 0\]
\[x(x) - x(200) = 0\]
\[x(x - 200) = 0\]

What are the solutions of the equation?

Problem 1

Modeling the Cable Shape

D. Interpret your solutions from part (C) in the problem situation. Use a complete sentence in your answer.
Investigate Problem 1

You have just solved the equation by factoring the polynomial. How do these solutions compare to the solutions from part (C)? Use a complete sentence in your answer.

Which method of finding the solutions was easier? Why? Use a complete sentence in your answer.

2. Just the Math: Factoring Out a Common Factor

In Question 1, you solved the equation by factoring out a common factor of $x$. Which mathematical rule was used to factor the polynomial? Use a complete sentence to explain your reasoning.

So, because the equality is symmetric, you can write your multiplicative distributive properties as follows:

\begin{align*}
ab + ac &= a(b + c) \\
ab - ac &= a(b - c) \\
ba + ca &= (b + c)a \\
ba - ca &= (b - c)a
\end{align*}

When the distributive properties are used in this manner, you are factoring expressions.

Factor each expression completely. Show your work.

\begin{align*}
x^2 + 12x \\
x^3 - 5x \\
3x^2 - 9x \\
10x^2 - 6
\end{align*}

Take Note

Recall the Symmetric Property from Lesson 4.5: For any real numbers $a$ and $b$, if $a = b$, then $b = a$.

Take Note

An expression is factored completely when none of the factors of the expression can be factored further.

3. What is the domain of the function in Problem 1? Use complete sentences to explain your reasoning.
Investigate Problem 2

1. If we want to factor the polynomial from part (A) as the product of two linear expressions \((x + a)\) and \((x + b)\), what must be true about the product \(ab\)? Explain your reasoning. Use a complete sentence in your answer.

What must be true about the sum \(a + b\)? Explain your reasoning. Use a complete sentence in your answer.

Make a list of pairs of negative integers whose product is 7500. Then find the sum of each pair of numbers. Stop when you find the pair of numbers that has a sum of –200.
Investigate Problem 2

Which pair of numbers has a product of 7500 and a sum of –200?

Complete the factorization of \(x^2 - 200x + 7500\).

\[x^2 - 200x + 7500 = (x + \underline{ }) (x + \underline{ })\]

What are the solutions of the equation \(x^2 - 200x + 7500 = 0\)? Show your work.

What is the cable’s horizontal distance from the left tower at 81 meters above high water level? Use a complete sentence in your answer.

2. Factor each trinomial as a product of linear factors. Then use the FOIL pattern to verify your answer. Show your work.

\[
\begin{align*}
x^2 + 4x + 3 & \quad x^2 - 4x - 5 \\
x^2 - x - 6 & \quad x^2 - 9x + 20 \\
x^2 + 18x + 81 & \quad x^2 - 36
\end{align*}
\]

3. How does the sign of the constant in the trinomial determine the signs of the constants in the linear factors? Use complete sentences in your answer.

Take Note

Your special product rules from Lesson 10.4 can help you factor special products:

\[
\begin{align*}
a^2 + 2ab + b^2 &= (a + b)^2 \\
a^2 - 2ab + b^2 &= (a - b)^2 \\
a^2 - b^2 &= (a + b)(a - b)
\end{align*}
\]
4. In Questions 1 and 2, what do all the trinomials have in common? How are they different? Use complete sentences in your answer.

5. Consider the trinomial $2x^2 + 11x + 9$. If we want to factor this expression as a product of linear factors, what must be true about the coefficients of the $x$-terms of the linear factors? Use complete sentences to explain your reasoning.

What must be true about the constants of the linear factors? Use complete sentences to explain your reasoning.

Now that you have identified all the possible coefficients of the $x$-terms and the constants for the linear factors, list the possible pairs of linear factors below that could result from the different positions of the numbers in the factors. The first one is listed.

$$(2x + 3)(1x + 3) = \text{ }$$

Now find the product of each pair of linear factors and write your answer above. Which pair of linear factors gives the factorization of $2x^2 + 11x + 9$?

6. Factor each trinomial as a product of linear factors. Then use the FOIL pattern to verify your answer. Show your work.

$$3x^2 + 7x + 2 = \text{ }$$

$$5x^2 + 17x + 6 = \text{ }$$

$$6x^2 + 19x + 10 = \text{ }$$
SCENARIO  A swimming pool company designs and installs custom in-ground swimming pools. A swimming pool design is based on the location, the amount of area available, and client preferences. In-ground swimming pools can have any shape and can be covered in tiles, fabric, or a concrete coating.

Problem 1  Pool Length and Area

A. The company’s designer is studying how the change in the length of a pool affects the change in the area of the top surface of the pool. A rectangular pool that is being designed is 20 feet longer than it is wide. Write expressions for the width and the length of the pool. Use the variable \( x \) to represent the width of the pool.

B. Write a proportion that compares the length of the pool to the area of the surface of the pool.

C. What is the domain of the expression in part (B)? Do not consider the problem situation when finding the domain. Use complete sentences to explain your reasoning.

D. How are the numerator and denominator of your expression the same? How are they different? Use complete sentences in your answer.

E. Just the Math: Rational Expressions  The proportion that you wrote in part (B) is a rational expression. A rational expression is a fraction in which the numerator and denominator are both polynomials. What are the polynomials, in standard form, of your rational expression in part (B)? What is the degree of each polynomial? Use a complete sentence in your answer.
1. Use your work from Problem 1 to describe how to find the domain of any rational expression. Use a complete sentence in your answer.

2. What is the form of the polynomials in your rational expression in part (B)? Use a complete sentence in your answer.

3. Do you think that it is easier to identify the domain of a rational expression when the numerator and denominator are in standard form or in factored form? Use a complete sentence to explain your reasoning.

4. Factor the numerator and denominator of each rational expression, if possible. Then identify the domain of the rational expression.

\[
\frac{4}{x + 1} \quad \frac{x + 3}{x^2 - 5x}
\]

\[
\frac{x - 3}{x^2 + 5x + 4} \quad \frac{x - 1}{x^2 + x - 6}
\]

5. Can you simplify your rational expression in part (B)? If so, what is your simplified expression? Explain how you found your answer. Use complete sentences in your answer.
6. Complete the table of values that compares your expression from part (B) to your expression from Question 5.

<table>
<thead>
<tr>
<th>Expression</th>
<th>$\frac{x + 20}{x(x + 20)}$</th>
<th>$\frac{1}{x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-25$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-20$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-10$</td>
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<td></td>
</tr>
<tr>
<td>$0$</td>
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</tr>
<tr>
<td>$10$</td>
<td></td>
<td></td>
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<tr>
<td>$20$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What do you notice about the values in the table? Use a complete sentence in your answer.

7. What is the domain of your simplified expression in Question 5? Is this domain different from the domain in part (C)? If so, how is it different? Use complete sentences in your answer.

In order for the original expression and the simplified expression to be equivalent, we must restrict the domain of the simplified expression so that it is the same as the original expression.

What value must be excluded from the domain of $\frac{1}{x}$ so that its domain is the same as the domain of $\frac{x + 20}{x(x + 20)}$?

We indicate that the expressions are equal by writing

$$\frac{x + 20}{x(x + 20)} = \frac{1}{x}, x \neq -20.$$
8. Simplify each rational expression. Then restrict the domain so that the simplified expression is equivalent to the original expression. Show your work.

\[
\frac{x}{x^2 - 3x} \quad \frac{x + 2}{x^2 + 3x + 2}
\]

\[
\frac{x - 6}{2x^2 - 11x - 6} \quad \frac{x^2 - 1}{x + 1}
\]

\[
\frac{x^2 + x}{x^2 - 6x - 7} \quad \frac{x^2 - 5x}{x^2 - 25}
\]

9. Just the Math: Multiplying Rational Expressions
You multiply rational expressions in the same way that you multiply fractions of numbers. For instance, to find the product of \(\frac{x + 1}{x - 3}\) and \(\frac{x}{x + 1}\), you multiply numerators and you multiply denominators, and then simplify. Complete the steps below to find the product.

\[
\frac{x + 1}{x - 3} \cdot \frac{x}{x + 1} = \left(\frac{\text{numerator: } x + 1}{\text{denominator: } x - 3}\right) \left(\frac{\text{numerator: } x}{\text{denominator: } x + 1}\right)
\]

Multiply numerators and multiply denominators.

\[
= \frac{\text{numerator: } x + 1}{\text{denominator: } x - 3}, \quad x \neq -1
\]

Simplify.

10. Just the Math: Dividing Rational Expressions
You divide rational expressions in the same way that you divide fractions of numbers. For instance, to find the quotient of \(\frac{x}{x - 5}\) and \(\frac{x^2}{x + 2}\), you multiply the dividend by the reciprocal of the divisor. Complete the steps below to find the quotient.

\[
\frac{x}{x - 5} \div \frac{x^2}{x + 2} = \frac{x}{x - 5} \cdot \frac{\text{numerator: } x}{\text{denominator: } x - 5}
\]

Multiply by reciprocal of divisor.

\[
= \left(\frac{\text{numerator: } x}{\text{denominator: } x - 5}\right) \left(\frac{\text{numerator: } x}{\text{denominator: } x - 5}\right)
\]

Multiply numerators and multiply denominators.

\[
= \frac{\text{numerator: } x}{\text{denominator: } x - 5}, \quad x \neq -2
\]

Simplify.
Investigate Problem 1

11. Find the product or quotient. Show your work and leave your answer in simplified form. Remember to restrict the domain, if necessary.

\[
\frac{x + 6}{x} \cdot \frac{x}{x - 5} \quad \frac{x + 3}{x - 2} \div \frac{x}{x - 4}
\]

\[
\frac{x^2 + 3x}{x - 4} \cdot \frac{x + 5}{x + 3}
\]

\[
\frac{x - 1}{x^2 - 4x} \div \frac{x + 1}{x}
\]

\[
\frac{x^2 - 7x}{x + 3} \cdot \frac{x^2 + 4x + 3}{x - 7}
\]

12. Do you think that it matters whether you factor your expressions before you multiply or divide? Use complete sentences to explain your reasoning.
Two workers are responsible for cutting and placing steel reinforcement bars (rebar) on the bottom and sides of a standard size pool.

A. It takes one of the workers nine hours to cut and place the rebar for a standard-sized pool by himself. Write a fraction that represents the amount of the job that is completed by this worker in one hour if the worker is doing the job by himself.

B. It takes the other worker an unknown number of hours to cut and place the rebar for a standard-sized pool by herself. Write a fraction that represents the amount of the job that is completed by this worker in one hour if the worker is doing the job by herself. Use $t$ to be the number of hours it takes this worker to complete the job by herself.

C. Use your answers from parts (A) and (B) to write an expression that shows the amount of the job that is completed in one hour by the workers if they are doing the job together.

D. Suppose that it takes the second worker eight hours to complete the job by herself. If the workers work together, what fraction of the job is completed in one hour? Show your work and use a complete sentence in your answer.

E. Use complete sentences to explain the steps that you took to find your answer to part (D).
Investigate Problem 2

1. You can add and subtract rational expressions in the same way that you add and subtract fractions. Consider your sum in part (C). Before you can add numerators, what must you do first? Use a complete sentence in your answer.

What is the least common denominator (LCD) of the rational expressions in part (C)? Use a complete sentence to explain how you found your answer.

By what expression do you have to multiply \( \frac{1}{t} \) so that the denominator is the LCD?

By what expression do you have to multiply \( \frac{1}{9} \) so that the denominator is the LCD?

Simplify the sum from part (C) by completing each step below.

\[
\frac{1}{9} + \frac{1}{t} = \frac{1}{9} \cdot \boxed{\phantom{1}} + \frac{1}{t} \cdot \boxed{\phantom{1}}
\]

Multiply each expression by an appropriate form of 1.

\[
\quad = \boxed{\phantom{1}} + \boxed{\phantom{1}}
\]

Multiply.

\[
\quad = \boxed{\phantom{1}}
\]

Add.

2. Find each sum or difference. Show your work and write your answer in simplified form.

\[
\frac{4}{x} + \frac{2}{3x} \quad \frac{1}{3} - \frac{8}{x - 2}
\]